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A New Error Analysis for a Cubic Spline Approximate Solution of a Class of Volterra Integro-Differential Equations

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Abstract. In this paper a third-order numerical method is considered which utilizes a twice continuously differentiable third degree spline to approximate the solution of

$$\dot{x}(t) = F\left(t, x(t), \int_a^t K(t, u, x(u))\, du\right),$$

$$x(a) = x_0,$$

at discrete points in the interval $[a, b]$. The error analysis uses a technique usually associated with linear multistep methods.

I. Introduction. In this paper, consideration is directed to the Volterra integro-differential equation

$$(1) \qquad \dot{x}(t) = F\left(t, x(t), \int_a^t K(t, u, x(u))\, du\right), \qquad a \leqq t \leqq b,$$

with the initial condition $x(a) = x_0$. A third-order numerical method is considered which utilizes a twice continuously differentiable third degree spline to approximate the solution x at discrete points in the interval $[a, b]$.

Other authors, e.g. Hung [5], have applied cubic splines to obtain an approximate solution of a scalar Volterra integro-differential equation. This paper considers the method as applied to vector equations. More important, however, is the error analysis presented herein. This analysis uses a lemma usually associated with linear multistep methods. The utilization of this lemma allows the cubic spline method to be applied to a larger class of equations than considered by Hung with, however, a corresponding reduction in the order of the errors. In particular, Hung requires the solution of (1) be of class $C^6[a, b]$ while the analysis presented here requires only $C^4[a, b]$. Accordingly, Hung achieves a discretization error $O(h^4)$ while this analysis achieves $O(h^3)$.

II. Notation and Assumptions. Let $R(F)$ and $R(K)$ be the regions defined by

$$R(F) = \{(t, x, y) : a \leqq t \leqq b; x, y \in E^n\}$$

and

$$R(K) = \{(t, u, y) : a \leqq u \leqq t \leqq b; y \in E^n\},$$

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where E^n is real Euclidean n-space. Moreover, let the nth order matrices $F^{(2)}$, $F^{(3)}$, and $K^{(3)}$ be such that the respective elements are given by

$$(2)\qquad F^{(2)}_{i,j} = \frac{\partial F_i}{\partial x_j}, \qquad F^{(3)}_{i,j} = \frac{\partial F_i}{\partial y_j}, \quad \text{and} \quad K^{(3)}_{i,j} = \frac{\partial K_i}{\partial y_j}.$$

Then, the following assumptions are made:

(a) Equation (1) has a unique solution.

(b) F and K are continuous mappings of $R(F)$ and $R(K)$ to E^n, respectively.

(c) The matrix elements (2) are continuous and bounded.

Assumption (c) has two important implications. First, there exist constants $\bar{F}^{(2)}$, $\bar{F}^{(3)}$ and $\bar{K}^{(3)}$ such that $\|F^{(2)}\| \leqq \bar{F}^{(2)}$, $\|F^{(3)}\| \leqq \bar{F}^{(3)}$ and $\|K^{(3)}\| \leqq \bar{K}^{(3)}$, where $\|\cdot\|$ will be used interchangeably to denote compatible matrix and vector norms. Secondly, Buck [1, p. 268], for $(t, x, y), (t, \bar{x}, y) \in R(F)$ there exist $p_i \in E^n$, $i = 1, \cdots, n$, such that

$$F(t, x, y) - F(t, \bar{x}, y) = F^{(2)}(x - \bar{x})$$

where $F^{(2)} = (F^{(2)}_{i,j}(t, p_i, y))$. Similarly, for $(t, x, y), (t, x, \bar{y}) \in R(F)$ and for (t, u, v), $(t, u, \bar{v}) \in R(K)$, there are $q_i, r_i \in E^n$, $i = 1, \cdots, n$, such that

$$F(t, x, y) - F(t, x, \bar{y}) = F^{(3)}(y - \bar{y})$$

and

$$K(t, u, v) - K(t, u, \bar{v}) = K^{(3)}(v - \bar{v})$$

where $F^{(3)} = (F^{(3)}_{i,j}(t, x, q_i))$ and $K^{(3)} = (K^{(3)}_{i,j}(t, u, r_i))$.

III. The Method. Let $[a, b]$ be divided into N equal subintervals of length $h = (b - a)/N$ with endpoints $t_0, t_1, \cdots, t_N$, called nodes. Let x_k, $\dot{x}_k$ and $\ddot{x}_k$ denote approximations for $x(t_k)$, $\dot{x}(t_k)$ and $\ddot{x}(t_k)$, respectively. The n-dimensional cubic spline S on $[a, b]$ is defined as follows: For $t \in [t_k, t_{k+1}]$, S is denoted by S_k and is defined by

$$(3)\qquad S_k(t) = x_k + (t - t_k)\dot{x}_k + \frac{(t - t_k)^2}{2}\ddot{x}_k + \frac{(t - t_k)^3}{3h^2}(\dot{x}_{k+1} - \dot{x}_k - h\ddot{x}_k).$$

Note that $S_k(t_k) = x_k$, $\dot{S}_k(t_k) = \dot{x}_k$, $\ddot{S}_k(t_k) = \ddot{x}_k$ and $\dot{S}_k(t_{k+1}) = \dot{x}_{k+1}$.

The approximate solution to (1) is obtained by replacing the integral by a numerical quadrature formula and requiring that the resulting equation be satisfied at the nodes. Thus, if the cubic spline S replaces x in this equation, (1) is replaced by

$$(4)\qquad \dot{S}_k(t_{k+1}) = F\left(t_{k+1}, S_k(t_{k+1}), h \sum_{i=0}^{k+1} w_i K(t_{k+1}, t_i, S_{i-1}(t_i))\right)$$

where the weights w_i are bounded and depend on the numerical quadrature formula used and where $S_{-1}(t_0) \equiv x_0$. Then, using $x_k = S_{k-1}(t_k)$, $\dot{x}_k = \dot{S}_{k-1}(t_k)$, $\ddot{x}_k = \ddot{S}_{k-1}(t_k)$ and $\dot{x}_{k+1} = \dot{S}_k(t_{k+1})$, (4) becomes

$$(5)\qquad \dot{x}_{k+1} = H(\dot{x}_{k+1})$$

where

$$H(\dot{x}_{k+1}) = F\left(t_{k+1}, x_k + h\dot{x}_k + \frac{h^2}{2}\ddot{x}_k + \frac{h}{3}(\dot{x}_{k+1} - \dot{x}_k - h\ddot{x}_k), q_{k+1}\right)$$

with

$$q_{k+1} = h\sum_{i=0}^{k} w_i K(t_{k+1}, t_i, x_i)$$
$$+ hw_{k+1}K\left(t_{k+1}, t_{k+1}, x_k + h\dot{x}_k + \frac{h^2}{2}\ddot{x}_k + \frac{h}{3}(\dot{x}_{k+1} - \dot{x}_k - h\ddot{x}_k)\right).$$

All quantities in (5) are known except $\dot{x}_{k+1}$. Since $\dot{x}_{k+1}$ determines S_k, the values $x_{k+1} = S_k(t_{k+1})$ and $\ddot{x}_{k+1} = \ddot{S}_k(t_{k+1})$ follow. (Although (5) is used to determine x_k, it is convenient to use (4) in the error analysis to follow.)

It follows, in the usual straightforward manner, from assumption (c) that, for x, $\bar{x} \in E^n$,

$$||H(x) - H(\bar{x})|| \leqq \frac{h\bar{F}^{(2)} + h^2\,|w_{k+1}|\,\bar{F}^{(3)}\,\bar{K}^{(3)}}{3}\,||x - \bar{x}||.$$

Thus, for h sufficiently small the mapping given by (5) is a contraction. This proves the following theorem.

THEOREM 1. *For H as defined by* (5) *and with assumption* (c) *satisfied, it follows that, for sufficiently small h, H is a contraction mapping.*

Thus, (5) can be used iteratively to determine x_i, $i = r, \cdots, N$, where r depends on the starting method used.

IV. Error Analysis. Let $E(t) = x(t) - S(t)$. Then, from (1) and (4), there follows

$$\dot{E}(t_k) = F\left(t_k, x(t_k), \int_{t_0}^{t_k} K(t_k, u, x(u))\,du\right) - F\left(t_k, S_{k-1}(t_k), \int_{t_0}^{t_k} K(t_k, u, x(u))\,du\right)$$
$$+ F\left(t_k, S_{k-1}(t_k), \int_{t_0}^{t_k} K(t_k, u, x(u))\,du\right)$$
$$- F\left(t_k, S_{k-1}(t_k), h\sum_{i=0}^{k} w_i K(t_k, t_i, x(t_i))\right)$$
$$+ F\left(t_k, S_{k-1}(t_k), h\sum_{i=0}^{k} w_i K(t_k, t_i, x(t_i))\right)$$
$$- F\left(t_k, S_{k-1}(t_k), h\sum_{i=0}^{k} w_i K(t_k, t_i, S_{i-1}(t_i))\right).$$

Thus, in view of assumption (c),

$$\dot{E}(t_k) = F^{(2)}_{(k)}E(t_k) + \tilde{O}(h^p) + hF^{(3)}_{(k)}\sum_{i=0}^{k} w_i K^{(3)}_{(k,i)}E(t_i) \tag{6}$$

where $F^{(2)}_{(k)}$ and $F^{(3)}_{(k)}$ indicate the matrices $F^{(2)}$ and $F^{(3)}$ depend on the index k and $K^{(3)}_{(k,i)}$ indicates the matrix depends on the indices k and i. Furthermore, the numerical quadrature formula is assumed to be such that

$$\int_{t_0}^{t_k} K(t_k, u, x(u))\, du - h \sum_{i=0}^{k} w_i K(t_k, t_i, x(t_i)) = \tilde{O}(h^\nu)$$

where $\tilde{O}(h^\nu)$ is a vector with components all $O(h^\nu)$.

The error analysis development is to obtain an equation involving E and $\dot{E}$ at the nodes. Then, (6) is used to provide an equation in E only. The error information at the nodes is then used to obtain error bounds at nonnodal points.

To proceed with the error analysis at the nodes, it is assumed the solution $x \in C^{(4)}[a, b]$. Then, for $t \in [t_k, t_{k+1}]$,

$$E(t) = E(t_k) + (t - t_k)\dot{E}(t_k) + \frac{(t - t_k)^2}{2} \ddot{E}(t_k) + \frac{(t - t_k)^3}{6} \dddot{E}(t_k) + \tilde{O}(h^4), \tag{7}$$

$$\dot{E}(t) = \dot{E}(t_k) + (t - t_k)\ddot{E}(t_k) + \frac{(t - t_k)^2}{2} \dddot{E}(t_k) + \tilde{O}(h^3), \tag{8}$$

and

$$\ddot{E}(t) = \ddot{E}(t_k) + (t - t_k)\dddot{E}(t_k) + \tilde{O}(h^2). \tag{9}$$

(Since $S^{(4)} = 0$, the error terms involve only the solution x and not the spline S.)

Evaluation of (7) and (8) at t_{k+1} and elimination of $\dddot{E}(t_k)$ provides

$$E(t_{k+1}) = E(t_k) + \frac{2h}{3} \dot{E}(t_k) + \frac{h}{3} \dot{E}(t_{k+1}) + \frac{h^2}{6} \ddot{E}(t_k) + \tilde{O}(h^4) \tag{10}$$

while elimination of $\ddot{E}(t_k)$ provides

$$\frac{h^3}{12} \dddot{E}(t_k) = E(t_k) - E(t_{k+1}) + \frac{h}{2} \dot{E}(t_k) + \frac{h}{2} \dot{E}(t_{k+1}) + \tilde{O}(h^4). \tag{11}$$

Evaluation of (9) at t_{k+1} and substitution of $\dddot{E}(t_k)$ from (11) yields

$$\ddot{E}(t_{k+1}) = \ddot{E}(t_k) + \frac{12}{h^2} E(t_k) - \frac{12}{h^2} E(t_{k-1}) + \frac{6}{h} \dot{E}(t_k) + \frac{6}{h} \dot{E}(t_{k+1}) + \tilde{O}(h^2). \tag{12}$$

Reduction of subscripts by one in (12) and substitution of the resulting expression for $\ddot{E}(t_{k-1})$ into the equation which results from the reduction of the subscripts by one in (10) yields

$$\frac{h^2}{6} \ddot{E}(t_k) = -E(t_k) + E(t_{k-1}) + \frac{2h}{3} \dot{E}(t_k) + \frac{h}{3} \dot{E}(t_{k-1}) + \tilde{O}(h^4),$$

which when substituted in (10) provides

$$E(t_{k+1}) - E(t_{k-1}) + \frac{h}{3} [\dot{E}(t_{k-1}) + 4\dot{E}(t_k) + \dot{E}(t_{k+1})] + \tilde{O}(h^4). \tag{13}$$

Finally, from (6) and (13), there follows

$$E(t_{k+1}) - E(t_{k-1}) = \frac{h}{3}[F^{(2)}_{(k-1)}E(t_{k-1}) + 4F^{(2)}_{(k)}E(t_k) + F^{(2)}_{(k+1)}E(t_{k+1})]$$
$$+ \frac{h^2}{3}\Bigg[F^{(3)}_{(k-1)}\sum_{i=0}^{k-1} w_i K^{(3)}_{(k-1,i)}E(t_i) + 4F^{(3)}_{(k)}\sum_{i=0}^{k} w_i K^{(3)}_{(k,i)}E(t_i)$$
$$(14) \qquad + F^{(3)}_{(k+1)}\sum_{i=0}^{k+1} w_i K^{(3)}_{(k+1,i)}E(t_i)\Bigg]$$
$$+ \tilde{O}(h^{\min(p+1,4)}).$$

In order to bound the discretization errors at the nodes, the following lemma is used, the proof of which is similar to that for Lemma 5.6 (Henrici, [3, p. 243]) and Linz's [7] lemma.

LEMMA. *Let $z_m \in E^n$, $m \geqq r$, be the solution of*

$$\rho_k z_{m+k} + \cdots + \rho_0 z_m = h(\beta_{k,m} z_{m+k} + \beta_{k-1,m+k-1} z_{m+k-1} + \cdots + \beta_{0,m} z_m)$$
$$+ h^2\left(\sum_{i=0}^{m+k} \mu_{m+k,i} z_i + \sum_{i=0}^{m+k-1} \mu_{m+k-1,i} z_i + \cdots + \sum_{i=0}^{m} \mu_{m,i} z_i\right) + \lambda_m$$

where all $\beta_{i,j}$ and $\mu_{i,j}$ are nth order matrices and the ρ_i are scalars. Assume the polynomial $\rho_k\xi^k + \rho_{k-1}\xi^{k-1} + \cdots + \rho_0$ satisfies the Dahlquist stability condition (Henrici [3, *p.* 218]). *Thus, if*

$$1/(\rho_k + \rho_{k-1}\xi + \cdots + \rho_0\xi^k) \equiv \gamma_0 + \gamma_1\xi + \cdots,$$

where $\rho_k \neq 0$, then $\Gamma \equiv \sup_i |\gamma_i| < \infty$ (Henrici [3, *p.* 242]). *Furthermore, assume $\|z_i\| \leqq Z$, $i = 0, 1, \cdots, k + r - 1$, and for all i, j, $\|\beta_{i,j}\| \leqq \beta$, $\|\mu_{i,j}\| \leqq \mu$, $\|\lambda_i\| \leqq \lambda$. Then, for sufficiently small h,*

$$\|z_n\| \leqq K^* e^{nhL^*}, \qquad n = 0, 1, \cdots, N,$$

where

$$K^* = [k\Gamma AZ + hb\mu^*\Gamma rZ + N\lambda\Gamma]/[1 - h\Gamma(\beta + b\mu^*)],$$
$$L^* = [\beta^*\Gamma + b\mu^*\Gamma]/[1 - h\Gamma(\beta + b\mu^*)],$$
$$A = |\rho_0| + \cdots + |\rho_n|,$$
$$\beta^* = \beta(k+1) \quad \textit{and} \quad \mu^* = \mu(k+1).$$

Application of the lemma to (14) yields

$$\|E(t_k)\| \leqq K^* e^{L^* t_k}$$

with

$$K^* = [4Z + 4hbw\bar{K}^{(3)}\bar{F}^{(3)}rZ + bO(h^{\min(3,p)})]/[1 - h(\tfrac{4}{3}\bar{F}^{(2)} + 4bw\bar{F}^{(3)}\bar{K}^{(3)})]$$

and

$$L^* = [4\bar{F}^{(2)} + 4bw\bar{F}^{(3)}\bar{K}^{(3)}]/[1 - h(\tfrac{4}{3}\bar{F}^{(2)} + 4bw\bar{F}^{(3)}\bar{K}^{(3)})]$$

where $|w_i| \leqq w$, $i = 0, \cdots, N$. Z and r depend on the starting method while p depends on the numerical integration method used. From K^* and L^*, it is readily seen that minimally the starting method should be $\tilde{O}(h^3)$ and $p = 3$. For this case $||E(t_k)|| = O(h^3)$. There follows from (6) and (10), respectively, that $||\dot{E}(t_k)|| = O(h^3)$ and $||\ddot{E}(t_k)|| = O(h)$. This proves the following theorem.

THEOREM 2. *If assumptions* (a), (b) *and* (c) *are satisfied,* $x \in C^{(4)}[a, b]$ *and the starting method and numerical integration method are both* $\tilde{O}(h^3)$, *then* $||E(t_k)|| = O(h^3)$, $||\dot{E}(t_k)|| = O(h^3)$ *and* $||\ddot{E}(t_k)|| = O(h)$.

The error analysis at nonnodal points proceeds by setting $t = t_{k+1}$ in (7) and (8), solving the resulting equations for $\ddot{E}(t_k)$ and $\dddot{E}(t_k)$ and substituting these equations back into (7) and (8) to obtain

$$\begin{aligned}E(t) = &\left[1 - \frac{3(t - t_k)^2}{h^2} + \frac{2(t - t_k)^3}{h^3}\right]E(t_k) + \left[\frac{3(t - t_k)^2}{h^2} - \frac{2(t - t_k)^3}{h^3}\right]E(t_{k+1})\\ &+ \left[(t - t_k) - \frac{2(t - t_k)^2}{h} + \frac{(t - t_k)^3}{h^2}\right]\dot{E}(t_k)\\ &+ \left[\frac{(t - t_k)^3}{h^2} - \frac{(t - t_k)^2}{h}\right]\dot{E}(t_{k+1}) + \tilde{O}(h^4)\end{aligned}$$

and

$$\begin{aligned}\dot{E}(t) = &\left[\frac{6(t - t_k)^2}{h^3} - \frac{6(t - t_k)}{h^2}\right]E(t_k) + \left[\frac{6(t - t_k)}{h^2} - \frac{6(t - t_k)^2}{h^3}\right]E(t_{k+1})\\ &+ \left[1 - \frac{4(t - t_k)}{h} + \frac{9(t - t_k)^2}{h^2}\right]\dot{E}(t_k) + \left[\frac{3(t - t_k)^2}{h^2} - \frac{2(t - t_k)}{h}\right]\dot{E}(t_{k+1})\\ &+ \tilde{O}(h^3).\end{aligned}$$

Hence, for $t_k \leqq t \leqq t_{k+1}$,

$$||E(t)|| \leqq ||E(t_k)|| + ||E(t_{k+1})|| + h\,||\dot{E}(t_k)|| + h\,||\dot{E}(t_{k+1})|| + O(h^4)$$

and

$$||\dot{E}(t)|| \leqq 6\left(\frac{1}{h}\,||E(t_k)|| + \frac{1}{h}\,||E(t_{k+1})|| + ||\dot{E}(t_k)|| + ||\dot{E}(t_{k+1})||\right) + O(h^3).$$

Thus, $||E(t)|| = O(h^3)$ and $||\dot{E}(t)|| = O(h^2)$. From (9) and (11) there follows $||\ddot{E}(t)|| = O(h)$. This proves the following theorem.

THEOREM 3. *If the conditions of Theorem* 2 *are satisfied, then, for* $t \in [a, b]$, t *a nonnodal point,* $||E(t)|| = O(h^3)$, $||\dot{E}(t)|| = O(h^2)$ *and* $||\ddot{E}(t)|| = O(h)$.

V. A Numerical Example. The scalar equation considered here is

$$\dot{x}(t) = -\frac{3}{16}(t - 1)x(t) + \int_0^t x(u)\,du + \frac{13}{3}(t - 1)^{10/3} + \frac{3}{16},$$

$$x(0) = -1,$$

which has the solution $x(t) = (t - 1)^{13/3}$. Gregory's third-order formula,

$$\int_{t_0}^{t_k} f \doteq h\left(\frac{5}{12}f_0 + \frac{13}{12}f_1 + f_2 + \cdots + f_{k-2} + \frac{13}{12}f_{k-1} + \frac{5}{12}f_k\right),$$

is used for numerical quadrature. In order to apply Gregory's formula, the values x_0, $\dot{x}_1$, x_2 and x_3 are needed. x_0 is known. To obtain the other starting values, let $G(t, x(t))$ represent the right side of the above equation, i.e., $\dot{x}(t) = G(t, x(t))$, and let $G_k = G(t_k, x(t_k))$. Then

$$x(t_{k+1}) = x(t_k) + \int_{t_k}^{t_{k+1}} G(t, x(t))\, dt$$

and

$$x(t_{k+2}) = x(t_k) + \int_{t_k}^{t_{k+2}} G(t, x(t))\, dt.$$

Thus, using Simpson's rule,

$$x_{k+1} = x_k + \frac{h}{6}(G_k + 4G_{k+1/2} + G_{k+1}) \tag{15}$$

and

$$x_{k+2} = x_k + \frac{h}{3}(G_k + 4G_{k+1} + G_{k+2}) \tag{16}$$

where $t_{k+1/2} = t_k + h/2$. The quadratic equation through the points (t_k, x_k), (t_{k+1}, x_{k+1}) and (t_{k+2}, x_{k+2}) evaluated at $t_{k+1/2}$ provides

$$x_{k+1/2} = \tfrac{3}{8}x_k + \tfrac{3}{4}x_{k+1} - \tfrac{1}{8}x_{k+2}, \tag{17}$$

and, similarly,

$$G_{k+1/2} = \tfrac{3}{8}G_k + \tfrac{3}{4}G_{k+1} - \tfrac{1}{8}G_{k+2}. \tag{18}$$

TABLE OF ERRORS

Step Size 2^{-P} P	$\lvert E(1.5)\rvert$	$\lvert \dot{E}(1.5)\rvert$	$\max \lvert E(t_i)\rvert$, $i = 1, 2, 3$ (*for starting method*)
2	$.865964 \times 10^{-3}$	$.170270 \times 10^{-2}$	$.636479 \times 10^{-2}$
3	$.915525 \times 10^{-3}$	$.101191 \times 10^{-2}$	$.423563 \times 10^{-3}$
	$(.108248 \times 10^{-3})$	$(.212838 \times 10^{-3})$	
4	$.227987 \times 10^{-3}$	$.218648 \times 10^{-3}$	$.274553 \times 10^{-4}$
	$(.114481 \times 10^{-3})$	$(.126889 \times 10^{-3})$	
5	$.380606 \times 10^{-4}$	$.352258 \times 10^{-4}$	$.174974 \times 10^{-5}$
	$(.284984 \times 10^{-4})$	$(.273310 \times 10^{-4})$	
6	$.545480 \times 10^{-5}$	$.498293 \times 10^{-5}$	$.110465 \times 10^{-6}$
	$(.475757 \times 10^{-5})$	$(.440322 \times 10^{-5})$	
7	$.728986 \times 10^{-6}$	$.662161 \times 10^{-6}$	$.693943 \times 10^{-8}$
	$(.681850 \times 10^{-6})$	$(.622616 \times 10^{-6})$	
8	$.941891 \times 10^{-7}$	$.853280 \times 10^{-7}$	$.434832 \times 10^{-9}$
	$(.911232 \times 10^{-7})$	$(.827701 \times 10^{-7})$	

Equations (15), (16), (17) and (18) are used iteratively to find x_1 and x_2, then x_3 and x_4. The value x_4 is not a needed starting value but it is obtained simultaneously with x_3. The method is used until the difference between two consecutive iterates does not exceed 10^{-12}. This starting method is $O(h^4)$.

Once the starting values are known, the method of Section III is used with Gregory's formula for the numerical integration. Note the method is applied to $\dot{x}_i$, not x_i. Once $\dot{x}_i$ is known, then

$$x_i = S_{i-1}(t_i) \quad \text{and} \quad \ddot{x}_i = \ddot{S}_{i-1}(t_i).$$

The above table summarizes the errors corresponding to various step sizes. According to the theory, if the step size is halved the errors in E and $\dot{E}$ should be reduced by approximately one-eighth. The numbers in parenthesis are one-eighth the error for the previous step size, i.e., the error predicted by the theory when the step size is halved.

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